~ Carleson's Counterexample ~

Rectangular BMO: $\|b\|_{BMO_r} := \sup_{R_0 \in \mathcal{R}} \left(\frac{1}{|R_0|} \sum_{\substack{R \subset R_0 \\ R \in \mathcal{R}}} (b, h_R)^2 \right)^{1/2}$

Product BMO: $\|b\|_{BMO_p} := \sup_{\Omega} \left(\frac{1}{|\Omega|} \sum_{\substack{R \subset \Omega \\ R \in \mathcal{R}}} (b, h_R)^2 \right)^{1/2}$ (Ω = open set w/ $|\Omega| < \infty$)

It is obvious that $BMO_r \supseteq BMO_p$, since $\|b\|_{BMO_r} \leq \|b\|_{BMO_p}$. The famous Carleson Counterexample shows that this inclusion is strict.

DEF.: A *quilt* is a finite collection $\mathcal{Q}$ of dyadic rectangles $R \in \mathcal{R}$, $R \subset [0,1]^2$ such that:

(1). $\sum_{R \in \mathcal{Q}} |R| = 1$

(2). $\sum_{\substack{R \in \mathcal{Q}: \\ R \subset Q}} |R| \leq |Q|$, for any dyadic rectangle $Q \in \mathcal{R}$, $Q \subset [0,1]^2$.

Define the *area* of a quilt $\mathcal{Q}$ to be $\left| \bigcup_{R \in \mathcal{Q}} R \right|$.

(For example, $\{[0,1]^2\}$ is a quilt with area 1).

→ Given a quilt $\mathcal{Q}$, consider the function

$$b := \sum_{R \in \mathcal{Q}} \sqrt{|R|}\, h_R$$

$$\Rightarrow \frac{1}{|R_0|} \sum_{R \subset R_0} (b, h_R)^2 = \frac{1}{|R_0|} \sum_{\substack{R \in \mathcal{Q} \\ R \subset R_0}} |R| \leq 1 \text{ for all } R_0 \in \mathcal{R} \Rightarrow \boxed{\|b_{\mathcal{Q}}\|_{BMO_r} \leq 1}$$

(if $R_0 \subset [0,1]^2$, this is just (2). Otherwise, the quantity is non-zero iff $R_0 \in \mathcal{R}$ contains $[0,1]^2$, in which case $\frac{1}{|R_0|} \sum_{\substack{R \in \mathcal{Q} \\ R \subset R_0}} |R| \leq \frac{1}{1} \underbrace{\sum_{R \in \mathcal{Q}} |R|}_{1} = 1$.)

Let $\Omega := \bigcup_{R \in \mathcal{Q}} R$ (open set)

$$\Rightarrow \|b\|_{BMO_p} \geq \frac{1}{|\Omega|} \underbrace{\sum_{R \in \mathcal{Q}} |R|}_{1} \Rightarrow \boxed{\|b\|_{BMO_p} \geq \frac{1}{\text{area}(\mathcal{Q})}}$$

→ Show that: There exist quilts w/ arbitrarily small area.

From this, it follows that $BMO_r \not\supseteq BMO_p$: if the spaces were the same, it would have to hold that $\underbrace{\|b\|_{BMO_p}}_{\text{arbitrarily large}} \leq C \underbrace{\|b\|_{BMO_r}}_{1}$ $\forall b$, for some universal C.

(*) Main idea : Given a quilt $\mathcal{Q}$ with area σ, one can construct from it another quilt $\widetilde{\mathcal{Q}}$, with area $\left(\sigma - \frac{1}{4}\sigma^2\right)$.

$\Rightarrow$ Assuming this, start with $\mathcal{Q} := \{[0,1]^2\}$, $\sigma_0 = 1$, and iterate the construction to obtain a sequence $\{\mathcal{Q}_n\}_n$ of quilts w/ areas σ_n that satisfy

$$\sigma_{n+1} = \sigma_n - \frac{1}{4}\sigma_n^2$$

$\Rightarrow \{\sigma_n\}_n$ is a positive, decreasing sequence $\Rightarrow$ σ_n has a limit l

$\Rightarrow$ l must satisfy $\quad l = l - \frac{1}{4}l^2 \Rightarrow l=0$

Prove (*): Fix a quilt $\mathcal{Q}$ with area σ. Choose an integer N large enough so that the side lengths of any rectangle in $\mathcal{Q}$ are $\geq \frac{1}{2^N}$.

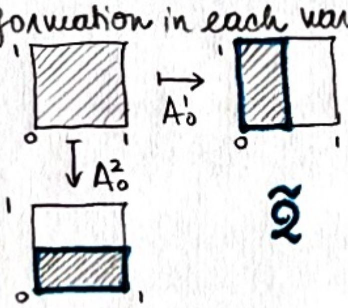
For every $\quad 0 \leq j < 2^N$, define the transformations A_j^1 and A_j^2 by:

$$\left.\begin{aligned} A_j^1(x,y) &:= \left(\frac{j}{2^N} + \frac{x}{2^{N+1}}, y\right) \\ A_j^2(x,y) &:= \left(x, \frac{j}{2^N} + \frac{y}{2^{N+1}}\right) \end{aligned}\right\} \qquad \boxed{\widetilde{\mathcal{Q}} := \left(\bigcup_{j=0}^{2^N-1} A_j^1(\mathcal{Q})\right) \cup \left(\bigcup_{j=0}^{2^N-1} A_j^2(\mathcal{Q})\right)}$$

Example: $\mathcal{Q} := \{[0,1]^2\} \Rightarrow$ Can take N to be $0, 1, \dots$

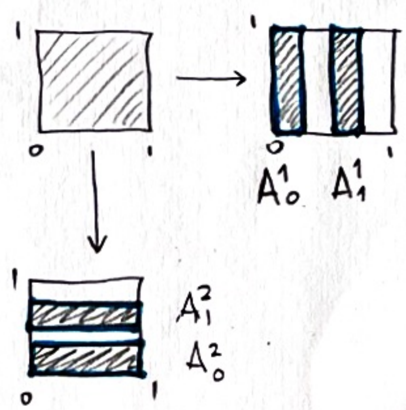
$N=0$: $\quad 0 \leq j < 1 \Rightarrow$ only one transformation in each variable :

$$A_0^1(x,y) = \left(\frac{x}{2}, y\right)$$
$$A_0^2(x,y) = \left(x, y/2\right)$$



$N=1$: $\quad 0 \leq j < 2 \Rightarrow$ two transformations in each variable :

$$A_j^1(x,y) = \left(\frac{j}{2} + \frac{x}{4}, y\right)$$
$$A_0^1(x,y) = \left(\frac{x}{4}, y\right)$$
$$A_1^1(x,y) = \left(\frac{1}{2} + \frac{x}{4}, y\right)$$



Show that $\widetilde{\mathcal{Q}}$ is a quilt:

- Clearly every $\widetilde{R} \in \widetilde{\mathcal{Q}}$ is in $[0,1]^2$;
- (1): $\sum_{\widetilde{R} \in \mathcal{Q}} |\widetilde{R}| = 1$

Any transformation A_j^i acting on some $R \in \mathcal{Q}$, $R = [a,b) \times [c,d)$ has
$$|\widetilde{R}| = |A_j^i(R)| = \frac{1}{2^{N+1}} |R|$$

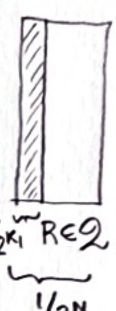
$$\left[\begin{array}{l} \text{for example, if } i=1: \\ [a,b) \mapsto \left[\frac{j}{2^N} + \frac{a}{2^{N+1}}, \frac{j}{2^N} + \frac{b}{2^{N+1}}\right) \\ \text{length} = \frac{1}{2^{N+1}}(b-a) \end{array}\right] \quad \begin{array}{l} \Rightarrow \sum_{R \in \mathcal{Q}} |A_j^i(R)| = \frac{1}{2^{N+1}} \underbrace{\sum_{R \in \mathcal{Q}} |R|}_{1} \\ \Rightarrow \sum_{\widetilde{R} \in \widetilde{\mathcal{Q}},\, \widetilde{R} \in A_j^i(\mathcal{Q})} |\widetilde{R}| = \frac{1}{2^{N+1}} \end{array}$$

There are 2^N of these rows & 2^N columns $\Rightarrow$ areas add up to $\underline{\underline{1}}$.

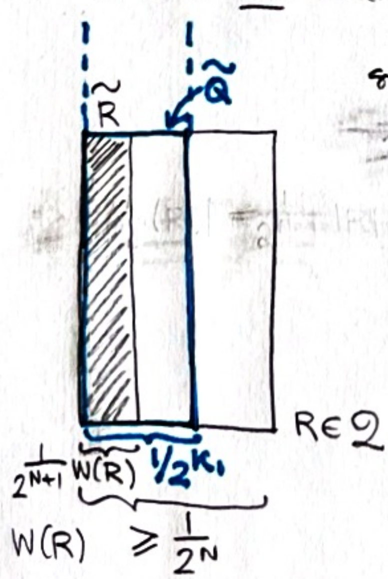
- (2): $\sum_{\substack{\widetilde{R} \in \widetilde{\mathcal{Q}}: \\ \widetilde{R} \subset \widetilde{Q}}} |\widetilde{R}| \leq |\widetilde{Q}|$, $\forall$ dyadic rectangle $\widetilde{Q} \subset [0,1]^2$.

Fix $\widetilde{Q} = I \times J$, with $|I| = \frac{1}{2^{K_1}}$ and $|J| = \frac{1}{2^{K_2}}$ $(K_1, K_2 > 0)$.

- $\boxed{K_1 > N}$ $\Rightarrow \frac{1}{2^{K_1}} < \frac{1}{2^N} \leq$ side length of any $R \in \mathcal{Q}$



$\Rightarrow$ at most $\underline{\text{one}}$ column $A_j^1(\mathcal{Q})$ can be contained in $\widetilde{Q}$, and $\underline{\text{no rows}}$



side length (width)

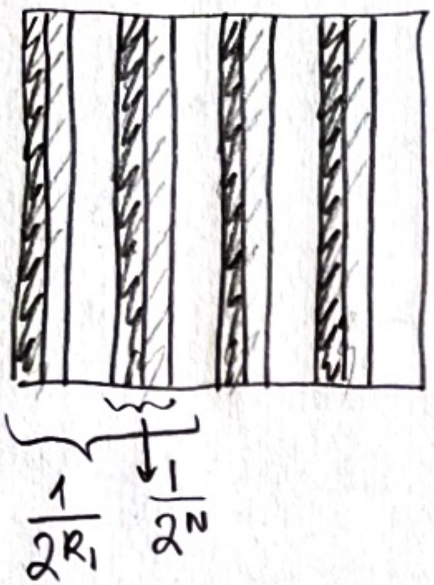
$$W(\widetilde{R}) = \frac{1}{2^{N+1}} W(R)$$

$\widetilde{Q}$ must have $W(\widetilde{Q}) = |I| = \frac{1}{2^{K_1}} \geq \frac{1}{2^{N+1}} W(R)$ and $h(\widetilde{Q}) \geq h(R)$

$$\Rightarrow |A_j^1(R)| = |\widetilde{R}| = \frac{1}{2^{N+1}} W(R) \cdot h(R) \leq \frac{1}{2^{K_1}} h(R) \leq |\widetilde{Q}|$$

- $\boxed{K_2 > N}$ Same

• $\boxed{R_1, R_2 \le N}$? => many rows/columns can contribute.



At most $\frac{1/2^{R_1}}{1/2^N} = \frac{2^N}{2^{R_1}}$ columns can contribute

Each column contributes at most $\frac{1}{2^{N+1}}|J|$

=> total column contribution $\le \frac{2^N}{2^{R_1}} \frac{1}{2^{N+1}} |J| = \frac{|I||J|}{2}$

$= \frac{1}{2}|\tilde{Q}|$

Similar analysis for rows

=> row contribution $\le \frac{1}{2}|\tilde{Q}|$

=> total contribution $\le |Q|$.

$$\left|\bigcup_{\tilde{R}\in\tilde{\mathcal{Q}}}\tilde{R}\right| = \sigma - \frac{1}{4}\sigma^2:$$

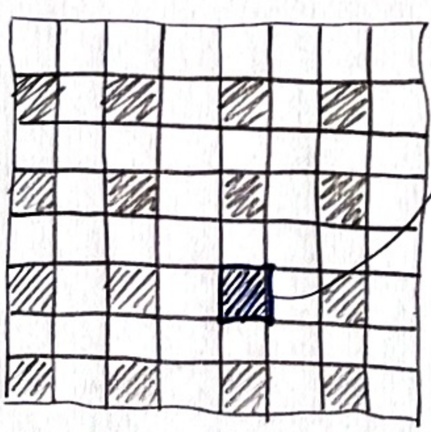
By construction: $\left|\bigcup_{j=0}^{2^N-1} A_j^1(\mathcal{Q})\right| = \sum_{j=0}^{2^N-1} \underbrace{\left|\bigcup_{\tilde{R}\in A_j^1(\mathcal{Q})}\tilde{R}\right|}_{\frac{1}{2^{N+1}}\left|\bigcup_{R\in\mathcal{Q}} R\right| = \frac{1}{2^{N+1}}\sigma} = 2^N\cdot\frac{1}{2^{N+1}}\sigma = \frac{1}{2}\sigma$

Similarly: $\left|\bigcup_{j=0}^{2^N-1} A_j^2(\mathcal{Q})\right| = \frac{1}{2}\sigma$

$$\Rightarrow \text{Area}(\tilde{\mathcal{Q}}) = \underbrace{\left|\bigcup_j A_j^1(\mathcal{Q})\right| + \left|\bigcup_j A_j^2(\mathcal{Q})\right|}_{\sigma} - \underbrace{\left|\left(\bigcup_j A_j^1(\mathcal{Q})\right)\cap\left(\bigcup_j A_j^2(\mathcal{Q})\right)\right|}_{\frac{1}{4}\sigma^2}$$

$$I_j := \left[\frac{j}{2^N}, \frac{j}{2^N}+\frac{1}{2^{N+1}}\right), \quad 0\le j<2^{N-1}.$$

$$\sum_{j_1=0}^{2^N-1}\sum_{j_2=0}^{2^N-1}\left|\left((I_{j_1}\times I_{j_2})\cap\bigcup_{\tilde{R}\in A_{j_1}^1(\mathcal{Q})}\tilde{R}\right)\cap\left((I_{j_1}\times I_{j_2})\cap\bigcup_{\tilde{R}\in A_{j_2}^2(\mathcal{Q})}\tilde{R}\right)\right| = \sum_{j_1,j_2}|E_{j_1\times j_2}|\,|F_{j_1\times j_2}| = \frac{\sigma^2}{4}$$



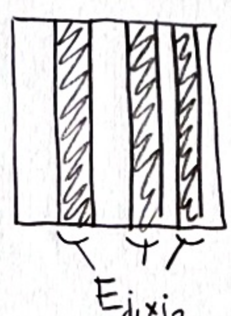
I_{j_2}

I_{j_1}

For a $I_{j_1}\times I_{j_2}$, $\exists$ one-dimensional sets $E_{j_1\times j_2}$ & $F_{j_1\times j_2}$ s.t.

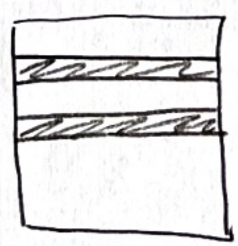
$$(I_{j_1}\times I_{j_2})\cap\left(\bigcup_{\tilde{R}\in A_{j_1}^1(\mathcal{Q})}\tilde{R}\right) = E_{j_1\times j_2}\times I_{j_2}$$

$$(I_{j_1}\times I_{j_2})\cap\left(\bigcup_{\tilde{R}\in A_{j_2}^2(\mathcal{Q})}\tilde{R}\right) = I_{j_1}\times F_{j_1\times j_2}$$



I_{j_2}

$E_{j_1\times j_2}$



$F_{j_1\times j_2}$

I_{j_1}

$$\sum_{j_2=0}^{2^N-1}|E_{j_1\times j_2}|\frac{1}{2^N} = \left|\bigcup_{\tilde{R}\in A_j^1(\mathcal{Q})}\tilde{R}\right| = \frac{\sigma}{2^{N+1}}$$

$$\Rightarrow \sum_{j_2=0}^{2^N-1}|E_{j_1\times j_2}| = \frac{\sigma}{2}$$

$$\Rightarrow \left|(E_{j_1\times j_2}\times I_{j_2})\cap(I_{j_1}\times F_{j_1\times j_2})\right| = |E_{j_1\times j_2}|\,|F_{j_1\times j_2}|$$

↳ doesn't depend on j_1 b/c all sides identical